

Symmetry breaking in noncommutative finite temperature $\lambda\phi^4$ theory with a nonuniform ground state

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Abstract

We consider the CJT effective action at finite temperature for a real scalar field theory with a nonuniform ground state which depends on the temperature. It turns out that to the first approximation the phase transition which breaks parity appears to be second order, and its critical temperature depends on the nonuniformity of the ground state.

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I. INTRODUCTION

Noncommutativity in field theory has received much attention in recent years, as a means to explore short distances, i.e. high energy effects [1]. In particular these effects could be due to a fundamental theory behind, which could be string theory [2]. In this direction, an interesting question which has been studied is the finite temperature behavior, in particular phase transitions [4, 5]. It turns out that there is a breaking of translational space-time symmetry in noncommutative theories, which in particular leads to the appearance of a stripe phase, related to the UV/IR mixing [3, 4, 6]. Further, in the study of noncommutative theories, the noncommutativity of time with space variables is known to lead to causality problems [7]. However, in the imaginary time formalism the space is euclidean and there is no reason why not to consider full noncommutativity [8]. In fact, the noncommutativity of the imaginary time could be related to noncommutative effects regarding temperature, which suggest the possibility of nonequilibrium processes. Such processes can be studied by means of the effective action of Cornwall, Jackiw and Tomboulis (CJT) [9, 10], which is given by a series of two-particle irreducible (2PI) Feynman diagrams and originally was proposed for composed particles [9]. In [11] the authors show that in a noncommutative $\lambda\phi^4$ field theory [12] with a constant ground state there is no symmetry breaking of the internal symmetry, however they show that in the stripe phase with space nonuniformity, symmetry can be broken. In this paper we consider the restauration of symmetry by thermal effects in the stripe phase including a dependence of the ground state on the imaginary time, hence on the temperature. In order to do it, following [11] we consider the CJT noncommutative effective action for a scalar theory with $\lambda\phi^4$ interaction, which we continue to imaginary time and then study its variations with respect to the degrees of freedom given by the field and the ‘propagator’. It turns out that there is a critical temperature which depends on the nonuniformity of the ground state, with a second order phase as a first approximation. The paper is organized as follows, in the second section we review some features of the CJT action, and in the third section we consider the finite temperature formulation, evaluate the critical temperature and show that the corresponding phase transition is first order. In the last section we draw some conclusions.

II. CJT EFFECTIVE ACTION

Let us consider an action of scalar fields $I(\Phi)$. The CJT effective action is obtained from the generating functional of the Green functions

$$Z(J, K) = \int D\Phi \exp \left\{ i \left[I(\Phi) + \int d^4x \Phi(x) J(x) + \frac{1}{2} \int d^4x d^4y \Phi(x) K(x, y) \Phi(y) \right] \right\} \quad (1)$$

by a double Legendre transformation of the generating functional of the connected diagrams $iW(J) = \log Z(J)$ [9],

$$\begin{aligned} \Gamma(\phi, G) = W(J, K) &- \int d^4x \phi(x) J(x) - \frac{1}{2} \int d^4x d^4y \phi(x) K(x, y) \phi(y) \\ &- \frac{1}{2} \int d^4x d^4y G(x, y) K(y, x) \end{aligned} \quad (2)$$

in such a way that

$$\frac{\delta W(J, K)}{\delta J(x)} = \phi(x), \quad (3)$$

$$\frac{\delta W(J, K)}{\delta K(x, y)} = \frac{1}{2} [\phi(x) \phi(y) + G(x, y)]. \quad (4)$$

Hence $\phi(x)$ is the vacuum expectation value of the field $\Phi(x)$, and $G(x, y)$ is the connected two point function. The physical solutions are obtained from the stationarity conditions,

$$\frac{\delta}{\delta \phi(x)} \Gamma(\phi, G) = 0, \quad (5)$$

$$\frac{\delta}{\delta G(x, y)} \Gamma(\phi, G) = 0. \quad (6)$$

The resulting effective action is [9],

$$\Gamma(\phi, G) = I(\phi) + \frac{i}{2} \text{Tr} \ln G^{-1} + \frac{i}{2} \text{Tr} [\Delta^{-1}(\phi) G] + \Gamma_2(\phi, G) - \frac{i}{2} \text{Tr}(1), \quad (7)$$

where,

$$iD^{-1}(x - y) = -(\partial^\mu \partial_\mu + m^2) \delta^4(x - y) \quad (8)$$

$$i\Delta^{-1}(x - y) = \frac{\delta^2 I(\phi)}{\delta \phi(x) \delta \phi(y)} = -(\partial^\mu \partial_\mu + m^2) \delta^4(x - y) + \frac{\delta^2 I_{\text{int}}(\phi)}{\delta \phi(x) \delta \phi(y)}. \quad (9)$$

and $\Gamma_2(\phi, G)$ is the 2PI diagrams expansion.

For a $\lambda \Phi^4$ action

$$I(\Phi) = \int_{-\infty}^{\infty} d^4x \left(\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi^4 \right), \quad (10)$$

Γ_2 is given to first order in λ by [9],

$$\Gamma_2(\phi, G) = -\frac{3\lambda}{4} \int_{-\infty}^{\infty} d^4x G^2(x, x). \quad (11)$$

III. NONCOMMUTATIVE FINITE TEMPERATURE ACTION

Let us consider the noncommutative action for a real scalar field with a $\lambda\Phi^4$ potential,

$$I(\Phi) = \int_{-\infty}^{\infty} d^4x \left(\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi * \Phi * \Phi * \Phi \right). \quad (12)$$

The noncommutative Weyl-Moyal product is given by,

$$\Phi(x) * \Psi(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu^x \partial_\nu^y} \Phi(x) \Psi(y) \Big|_{y=x} \quad (13)$$

We are interested in noncommutative finite temperature field theory. The imaginary time formulation is obtained from the relativistic field theory by setting the fields periodic in the imaginary time variable $\tau = it$, $\Phi(\tau + \beta) = \Phi(\tau)$, with an imaginary time integration interval $[0, \beta]$, setting $i \int d^4x \rightarrow \int_0^\beta d\tau \int d^3x$. The fourier transform of the fields is then given by

$$\phi(x) = \frac{1}{\beta} \sum_{-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} e^{i(\omega_n \tau + \vec{p}\vec{x})} \phi_n(p), \quad (14)$$

where $\omega_n = \frac{2\pi n}{\beta}$ is the so called Matsubara frequency, see e.g. [13]. Thus the free action entering into the CJT effective action will be,

$$\begin{aligned} I_0(\phi) &= -\frac{1}{2} \int_0^\beta d\tau \int d^3x [(\partial_\tau \phi)^2 + (\partial_i \phi)^2 + m^2 \phi^2] \\ &= -\frac{1}{2\beta} \sum_{-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} (p_n^2 + m^2) \phi_n(\vec{p}) \phi_{-n}(-\vec{p}), \end{aligned} \quad (15)$$

where $p_n \equiv (\omega_n, \vec{p})$, i.e. $p_n^2 = \omega_n^2 + \vec{p}^2$.

Further, we write the Weyl-Moyal product as

$$\phi(x) * \phi(x) = e^{\frac{i}{2} \partial_x \wedge \partial_y} \phi(x) \phi(y) \Big|_{y=x} \equiv e^{\frac{i}{2} [\Theta_{\tau i} (\partial_\tau^x \partial_i^y - \partial_i^x \partial_\tau^y) + \Theta_{ij} \partial_i^x \partial_j^y]} \phi(x) \phi(y) \Big|_{y=x}, \quad (16)$$

where $\Theta_{\tau i} = i\theta^{0i}$ and $\Theta_{ij} = \theta^{ij}$. Therefore, for the interaction action we get

$$I_{int}(\phi) = \frac{\lambda}{4!} (2\pi)^2 \prod_{k=1}^4 \left[\sum_{n_k} \int \frac{d^3p}{(2\pi)^3} \phi_{n_k}(\vec{p}_k) \right] e^{\frac{i}{2} (p_{n1} \wedge p_{n2} + p_{n3} \wedge p_{n4})} \delta_{\sum_i n_{i,0}} \delta^3 \left(\sum_i \vec{p}_i \right), \quad (17)$$

where

$$p_{n1} \wedge p_{n2} = \Theta_{\tau i} (\omega_{n1} p_{2i} - \omega_{n2} p_{1i}) + \Theta_{ij} p_{1i} p_{2j} = \vec{\Theta}_\tau (\omega_{n1} \vec{p}_2 - \omega_{n2} \vec{p}_1) + \vec{\Theta} (\vec{p}_1 \times \vec{p}_2) \quad (18)$$

where $(\Theta_\tau)_i = \Theta_{\tau i}$ and $\Theta_i = \epsilon_{ijk} \Theta_{jk}$.

The noncommutative CJT action can be obtained from (12) and (7), with the nonplanar contributions included by hand [11], following the definition of Γ . As mentioned in the introduction, we will consider a non translational invariant ground state field. In [4, 11], following [14], the field is chosen to be given by a stripe phase $\phi(x) = A \cos(\vec{Q}\vec{x})$, which does not depend on time in order to keep energy conserved. However, considering the possibility of non closed systems [15], we will take a dependence on the temperature as follows

$$\phi(\tau, \vec{x}) = A \cos(\omega_n \tau + \vec{Q}\vec{x}). \quad (19)$$

For simplicity, we keep $G(x, y) = G(x - y)$ translational invariant.

In order to get the finite temperature 2PI effective action, we write (7) as,

$$i\Gamma(\phi, G) = iI(\phi) - \frac{1}{2} \text{Tr} \ln G^{-1} + \frac{1}{2} \text{Tr} \left\{ \frac{\delta^2}{\delta\phi\delta\phi} [iI(\phi)] G \right\} + i\Gamma_2(\phi, G) + \frac{1}{2} \text{Tr}(1). \quad (20)$$

All terms in this expression, modulo additive constants, become real by the continuation to the imaginary time, and we get

$$\begin{aligned} i\Gamma(\phi, G) = & -\frac{1}{2\beta} \sum_m \int \frac{d^3p}{(2\pi)^3} (p_m^2 + m^2) \phi_m(\vec{p}) \phi_{-m}(-\vec{p}) \\ & + \frac{\lambda}{4!} (2\pi)^3 \prod_{k=1}^4 \left[\sum_{n_k} \int \frac{d^3p_k}{(2\pi)^3} \phi_{n_k}(\vec{p}_k) \right] e^{\frac{i}{2}(p_{n1} \wedge p_{n2} + p_{n3} \wedge p_{n4})} \delta_{\sum_i n_i, 0} \delta^3 \left(\sum_i \vec{p}_i \right) \\ & + \frac{1}{2} \sum_m \int d^3p \{ (p_m^2 + m^2) G(p_m) + \log[G(p_m)] \} \delta^3(\vec{0}) \\ & + \frac{\lambda}{4\beta} \sum_{l,m} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[\frac{1}{\beta} \phi_l(\vec{p}) \phi_{-l}(-\vec{p}) G(q_m) - \frac{(2\pi)^3}{2} G(p_l) G(q_m) \delta^3(\vec{0}) \right] \left[1 + \frac{1}{2} e^{i(p_l \wedge q_m)} \right] + \text{const.} \end{aligned} \quad (21)$$

Further, we compute the Fourier expansion coefficients of (19)

$$\phi_m(\vec{p}) = (2\pi)^3 \frac{\beta}{2} A \left[\delta_{mn} \delta^3(\vec{p} - \vec{Q}) + \delta_{m,-n} \delta^3(\vec{p} + \vec{Q}) \right], \quad (22)$$

from which we get,

$$\begin{aligned} i\Gamma(\phi, G) = & \left\{ -\frac{1}{2} (2\pi)^3 \beta A^2 \left(Q_n^2 + m^2 + \frac{\lambda}{8} A^2 \right) \right. \\ & + \frac{1}{2} \sum_m \int d^3q \{ (q_m^2 + m^2) G(q_m) + \log[G(q_m)] \} \\ & + \frac{\lambda}{8} A^2 \sum_m \int d^3q G(q_m) \left[1 + \frac{1}{2} \cos(Q_n \wedge q_m) \right] \\ & \left. - \frac{\lambda}{8} \frac{1}{(2\pi)^3 \beta} \sum_{l,m} \int d^3p d^3q G(p_l) G(q_m) \left[1 + \frac{1}{2} \cos(p_l \wedge q_m) \right] \right\} \delta^3(\vec{0}) + \text{const.} \end{aligned} \quad (23)$$

Thus, varying this effective action with respect to A and $G(p_k)$, we get respectively the equations

$$A \left\{ \frac{\lambda}{8} A^2 + Q_n^2 + m^2 - \frac{\lambda}{(2\pi)^3 \beta} \sum_m \int d^3 q G(q_m) \left[1 + \frac{1}{2} \cos(Q_n \wedge q_m) \right] \right\} = 0, \quad (24)$$

$$G^{-1}(p_k) + p_k^2 + m^2 + \frac{\lambda}{4} A^2 \left[1 + \frac{1}{2} \cos(Q_n \wedge p_k) \right] - \frac{\lambda}{2(2\pi)^3 \beta} \sum_m \int d^3 q G(q_m) \left[1 + \frac{1}{2} \cos(p_k \wedge q_m) \right] = 0. \quad (25)$$

From the second equation, we see that the integrals $\int d^3 q G(q_m)$ have an UV divergence and in order to regularize it, we make both a high-momentum cutoff [4] and a mass renormalization [11] by setting,

$$\mu^2 = m^2 - \frac{\lambda}{2(2\pi)^3 \beta} \sum_m \int \frac{d^3 q}{q_m^2 + \sigma^2}. \quad (26)$$

Further we define the integrals

$$I(\sigma) = - \sum_m \int d^3 q \left[G(q_m) + \frac{1}{q_m^2 + \sigma^2} \right], \quad (27)$$

$$I_1(p_k) = - \sum_m \int d^3 q G(q_m) \cos(p_k \wedge q_m), \quad (28)$$

$$I(p_k) = I(\sigma) + \frac{1}{2} I_1(p_k). \quad (29)$$

Hence we have from (25)

$$G^{-1}(p_k) = -p_k^2 - \mu^2 - \frac{\lambda}{4} A^2 \left[1 + \frac{1}{2} \cos(Q_n \wedge p_k) \right] - \frac{\lambda}{2(2\pi)^3 \beta} I(p_k). \quad (30)$$

Equation (24) has one solution $A = 0$, and the other solution is

$$\frac{\lambda}{8} A^2 = -Q_n^2 - \mu^2 - \frac{\lambda}{2(2\pi)^3 \beta} I(Q_n). \quad (31)$$

Which substituted into (30) gives

$$G^{-1}(p_k) = -p_k^2 - \mu^2 + 2(\mu^2 + Q_n^2) \left[1 + \frac{1}{2} \cos(Q_n \wedge p_k) \right] + \frac{\lambda}{2(2\pi)^3 \beta} \left\{ \left[1 + \frac{1}{2} \cos(Q_n \wedge p_k) \right] I(Q_n) - \frac{1}{2} I(p_k) \right\} \quad (32)$$

In order to see under which conditions (31) has solutions, we consider the integral (29)

$$\begin{aligned} I(Q_n) &= - \sum_m \int d^3 q \left\{ G(q_m) \left[1 + \frac{1}{2} \cos(Q_n \wedge q_m) \right] + \frac{1}{q_m^2 + \sigma^2} \right\} \\ &= \sum_m \int d^3 q \left\{ \frac{1 + \frac{1}{2} \cos(Q_n \wedge q_m)}{q_m^2 + \mu^2 - 2(\mu^2 + Q_n^2) \left[1 + \frac{1}{2} \cos(Q_n \wedge q_m) \right]} - \frac{1}{q_m^2 + \sigma^2} \right\} + \mathcal{O}(\lambda) \end{aligned} \quad (33)$$

The denominator of the first term on the r.h.s. is given by

$$\gamma_m(\vec{q}, T) = q_m^2 + \mu^2 - 2(\mu^2 + Q_n^2) \left[1 + \frac{1}{2} \cos(Q_n \wedge q_m) \right] \quad (34)$$

which is positive for a sufficiently high value of q_m^2 . Otherwise it can be negative, e.g. $\gamma_m(\vec{0}, 0) = -2\mu^2 - 3Q_n^2$. Hence (35) vanishes at some point, in particular when the temperature increases. Further, unless $\mu^2 < 0$, at low temperatures the r.h.s. of (31) is negative because the contribution of the integral (33) is suppressed by its factor λT . Thus, if we want to have solutions for (31), we must set $\mu^2 < 0$ and $|\mu|^2 > \vec{Q}^2$. Making explicit the temperature dependence we have,

$$\begin{aligned} \gamma_m(\vec{q}, T) = & (2\pi)^2 \left\{ m^2 - 2n^2 \left(1 + \frac{1}{2} \cos \left[2\pi \vec{\Theta}_\tau (n\vec{q} - m\vec{Q})T + \vec{\Theta}(\vec{Q} \times \vec{q}) \right] \right) \right\} T^2 \\ & + \vec{q}^2 - |\mu|^2 + 2(|\mu|^2 - \vec{Q}^2) \left\{ 1 + \frac{1}{2} \cos \left[2\pi \vec{\Theta}_\tau (n\vec{q} - m\vec{Q})T + \vec{\Theta}(\vec{Q} \times \vec{q}) \right] \right\} \end{aligned} \quad (35)$$

where the first part on the r.h.s., the one proportional to T^2 , is negative at least for $m = 0$. Further, we will consider small deviations from uniformity, hence we consider small values of \vec{Q}^2 . Then, if we start from $\gamma_m(\vec{0}, 0) = 2|\mu|^2 - 3\vec{Q}^2$ and make a small increase of the order δ in the temperature and \vec{q}^2 , to a first order we get

$$\begin{aligned} \gamma_m(\vec{q}, T) \simeq & \left(\frac{4}{3} - \delta \right) \left(|\mu|^2 - \frac{3}{2} \vec{Q}^2 \right) + \left(\vec{q}^2 - \frac{3}{2} \vec{Q}^2 \right) \delta + \frac{2}{3} \vec{q}^2 \\ & + (2\pi)^2 \left[\left(\frac{2}{3} + \delta \right) m^2 - n^2 \right] T^2 \end{aligned} \quad (36)$$

The temperature independent term is positive if $|\mu|^2 > \frac{3}{2} \vec{Q}^2$ and is least for $\vec{q}^2 = 0$. Further the temperature dependent term is negative at least for $m = 0, 1$ and is least for $m = 0$, considering that $n > 0$. Therefore under these assumptions, (35) will vanish when the temperature reaches a certain value, which is minimum for $m = 0$ and $\vec{q} = 0$, and in this case is given by

$$T_0 = \frac{2}{3(2\pi)^2} \left(|\mu|^2 - \frac{3}{2} \vec{Q}^2 \right). \quad (37)$$

Further, the numerator in the integrand in the first term of the r.h.s of (33) is positive. Hence at least to the lowest order in λ and for low temperatures the integral will be positive, and it will increase and diverge as T approaches T_0 . In fact, a numerical evaluation taking $n = 1$ and $m = 0$, shows that (35) decreases monotonically as the temperature rises and becomes zero at T_0 . Further, returning to (31) with the values of the preceding paragraph,

i.e. $n = 1$ and for relatively low values of $|\mu|^2$, a numerical evaluation of (33) shows that for a temperature interval $[0, T \lesssim T_0]$, up to a minus sign, the last term in (31) increases with temperature. Hence the critical temperature will be reached at a temperature $T_c < T_0$, which depends on noncommutativity by means of its dependence on the nonuniformity of the ground state. To the present approximation the phase transition is second order, in agreement with [4].

Finally, from (31) we get for $T < T_c$

$$A^2 = \frac{8}{\lambda} \left[|\mu|^2 - Q_n^2 - \frac{\lambda}{2(2\pi)^3 \beta} I^{(0)}(Q_n) \right] + \mathcal{O}(\lambda) = v^2, \quad (38)$$

which is a minimum because

$$i \frac{\delta^2 \Gamma}{\delta A^2} = -(2\pi)^3 \beta v^2. \quad (39)$$

Therefore, to the considered conditions and approximation, there are two nonvanishing solutions $A = \pm v$, with mass gap (39).

IV. CONCLUSIONS

In this work we have studied the noncommutative CJT effective action at finite temperature for a real scalar field theory with a nonuniform ground state field which depends on temperature. We take noncommutativity among all variables, including imaginary time, hence temperature is affected also. This fact hints to the possibility of nonequilibrium processes, for which the CJT 2PI effective action is a suitable formalism. Parity is broken at low temperatures and when the temperature rises it is restored. Under the approximations and considerations made, we have found a limiting value for this temperature and that this phase transition is second order, in accord with [4]. Furthermore, the critical temperature depends on the physical mass of the scalar field and on the wave vector of the nonuniform ground state field, hence it depends on noncommutativity implicitly, although we would expect an explicit dependence in a more detailed computation.

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